The Process of Teaching Mathematics by the Use of Non-traditional Formulations of Problems and by Use of Problems with Erroneous Decisions

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Abstract: *Some methodical ways in the teaching of mathematics in a mathematical-oriented high school are described. The basic techniques in which students are taught are through purposely incorrect definitions and theorems or by fallible methods of problem solutions. Also the problems with non-traditional formulations are considered. In particular there are problems with an applied intention from natural sciences, engineering, economics, etc. Using these described techniques students will learn the material better and also understand the various nuances more profoundly. It is very important that these methods develops complete independence of students, their analytical thinking, self-control and critical attitude to the stated material.*

Keywords: teaching methodology, erroneous decisions, self-control, non-traditional formulations.

1. Introduction

There is a gap in the Russian mathematical education between the knowledge level of graduating high school students and the requirements of universities, especially those in which mathematics is a specialized subject. Most first-year students are characterized by: inability to distinguish between what they understand and what they do not understand; inability to distinguish the correct verbal proof from the invalid one, the necessary conditions from sufficient ones; misconception of the dominant and unimportant, etc. (Kudryavtsev, 2008).

Specifically for removal of this gap in Russia 20-30 years ago, the academics began to organize the mathematics classes at leading universities on the base of "ordinary" school. University professors conduct teaching of mathematics, physics and computer science, while the school teachers undertake the teaching of other subjects. Training in these classes as rule lasts 2 or 3 years. After graduation, most graduating high school students enter into mathematics, physics, engineering or economic departments of universities.

The main purpose of pre-university education in school is to foster the general mathematical culture. The primary goal is not merely a set of techniques, methods and algorithms, but a profound and fundamental mathematical training by way of a systematic study of solution methods of carefully classified problems. The basic principle of learning is not to "know how", but to "know why".

With this in mind, academic activities are frequently or regularly used methods, instructive independence, unorthodoxy, unconventional thinking, self-control and critical quotient to the material.

2. The use of a purpose-designed incorrect or fallible definitions, theorems and solutions of the problems

One of the problems regarding new students is the different entry-level of training of school children. For example, when students are solving simple inequality $-\frac{1}{3}x < -2$, only one third of them is making it correctly by multiply by -3

or dividing by $-\frac{1}{3}$ with the changing the sign of inequality. The second third is making it correctly but it is too long. And finally the remaining students are solving this inequality incorrectly.

Thereby we observe somewhat different entry-level of students. This is due to the fact that before that school children studied in different schools, with different teachers, according to the different programs, revealing very different academic achievements. As a consequence, it is necessary to establish some kind of training phase, lasting from three to five months, which is reiterative to a large extent. The aim of this special training is to equalize the skills and abilities of the students thus reducing the gap between the superior and inferior students. It is of utmost importance that the lessons should be useful and interesting for all students, despite their level of abilities.

The method of repetition and correction of already covered material is proposed, based on specially designed solutions of problems with various errors, omissions and inaccuracies (Zelenskiy, 2012). This is not just an exercise of repetition, but the correction of the knowledge and skills, by which the more gifted students excel by reaffirming what they already know. It is beneficial for them to obtain something new for their knowledge base. The more challenged students slowly begin to understand their problems and rapidly catch up with the superior group.

It's necessary to note that many researchers were preoccupied with the issues surrounding mathematical errors, their typology and causes (Booker, 1988; Bradis, Minkovskii & Kharcheva, 1963; Gagatsis & Kyriakides, 2000; Newman, 1983; Perso, 1992; Radatz, 1979, etc.). However in this work the mathematical errors are not seen as a phenomenon which should be prevented and rejected without reservation. The attempt to draw benefit from this phenomenon is by way of the actual error, as it plays a major role as a learning mechanism.

Analysis of incorrect decision and bug scanning can bring enormous benefits. The question is about "ideological" errors, not just about arithmetic miscalculations. Two basic forms of work with erroneous decisions are applied. The teacher can simply demonstrate a "solution" of the problem on the board. At the same time he should be more convincing, by displaying some artistry. It often happens that students notice the trick and this is good. However it happens that the demonstration of "solution" is complete and all the students "understood" it with no questions asked. In such cases, it is important to take the audience from the "sleep" state and "blow up" the process in order to hint that the stated "solution" is not alright. And further analysis of the problem in this case is usually more useful to students than the "correct" solution.

The second form (more economical in time) lies in the fact that the teacher gives papers with selection of "solutions" on the subject usually as homework. The aim of students is to find errors and correct them. In the process of further examination in the class all errors are carefully analyzed. In addition, various approaches to the solution are discussed. By the example of these "solutions" students understand greater a particular method of solution and to identify any nuances.

By repeating theme we give to the students from 5 to 15 different tasks with erroneous or flawed decisions. For example, the equation $|x-1|+|x-2|=1$ is considered by repeating the theme "The equations and inequalities with absolute value". Students are invited to the following "solution".

We use the method of intervals. We examine the three gaps: a) $\begin{cases} x \le 1, \\ 1-x+2-x=1 \end{cases} \Leftrightarrow \begin{cases} x \le 1, \\ x=1 \end{cases} \Leftrightarrow x=1$;

b) $\begin{cases} 1<x<2, \\ x-1+2-x=1 \end{cases} \Leftrightarrow \begin{cases} 1<x<2, \\ 1=1. \end{cases} \Leftrightarrow x \in \varnothing$; c) $\begin{cases} x \ge 2, \\ x-1+x-2=1 \end{cases} \Leftrightarrow \begin{cases} x \ge 2, \\ x=2 \end{cases} \Leftrightarrow x=2$. Answer: 1; 2.

Students as a rule quickly discover an error in paragraph b) of this "solution". System $\begin{cases} 1<x<2, \\ 1=1 \end{cases}$ means, that where $x \in (1;\, 2)$ equation becomes an identity. It means that any value x from $x \in (1;\, 2)$ is performed. The answer on this gap will be $x \in (1;\, 2)$, and in the final response we must add points 1 and 2. It should be properly explained to the class.

It is noteworthy for the students that nothing fundamentally changes if we perform the decomposition into different intervals (except the case of the partition $(-\infty;1)\cup[1;2]\cup(2;+\infty)$, where decision will be a little more economical, which we could not assume before).

And it is necessary to offer the best way of finding the solution, based on the geometric sense of the absolute value. For the solution of our equation it is necessary to find such points x on the complete number scale, for which the sum of the distances from points 1 and 2 is equal to 1. It is clear all points of the segment $x\in[1;2]$ are satisfied to this condition, and to points outside this interval the sum of the distances will be greater than 1.

By this means, reviewing and commenting on this "solution", we repeated the main points of the method of intervals, examined some of the technical nuances and also illustrated the geometrical meaning of the module.

We can use this method for the learning new material too. For example, during treatment of theoretical material the teacher purposely provides an incorrect definition or wording of the theorem (often some important restriction is omitted). A classic example: inexact definition of a periodic function which occurs in a number of books: "Function with the property that $f(x+T)=f(x)$ for all x are called periodic function".

After the teacher gives an inexact "definition", the lesson can last for some time, until someone of the students discovers the need to add the definition of terms: a) $T>0$; b) x belongs to the domain of $f(x)$; c) $f(x-T)=f(x)$ and more precisely the requirement is that $x-T$ also belongs to the domain. It is the intent of the teacher to mislead the students to a contradiction in order to arrive at the correct definition. It is clearly that if the right definition was immediately given, many students would have missed important nuances.

During the proof of the theorem on the sum of the angles of a triangle it is very useful to give the following "proof" in the beginning: "Drop from the point B perpendicular BD to AC (fig. 1). Denoting the sum of the angles of a triangle in terms x, we obtain for the triangle ABD: $\alpha+\angle ABD+90°=x$, and for the triangle BDC: $\gamma+\angle DBC+90°=x$. Adding term by term these two equations, we obtain: $\alpha+\gamma+\angle ABD+\angle DBC+90°+\ 90°=2x$ or $(\alpha+\beta+\gamma)+180°=2x$. Therefore $x+180°=2x$, $x=180°$".

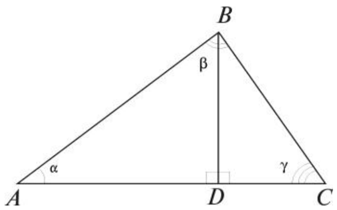

Figure 1.

This proof, of course, is much easier and clearer than those which are written in school textbooks. However it consists of vicious circle. In truth the theorem is established: "If the sum of the angles of all triangles is equal to one and the same number, this number is equal to $180°$". The trouble is that it is not easy to prove that the sum of the angles of a triangle is a constant. It is necessary to explain this to the students.

This method of training and repetition has many advantages:

1) The level of interest on behalf of the student regarding the described material is highly relevant, even when it seems to him that "he knows it";

2) As a result of detailed analysis of any defect, all students become focused on this heading and their knowledge becomes conscious. If the right definition was immediately given, some students could miss important nuances;

3) The class is constantly conducted under this "tone" so that the students respond and react to the phrases of the teacher;

4) Self-control and critical attitude towards the stated material are taught;

5) The student utilizes the necessary skills and algorithms for finding errors and defects in his own arguments and calculations;

6) The student is given the opportunity to learn from others' mistakes and to analyze and understand what someone has executed poorly and thus avoiding it;

7) Training for the search procedure of errors in the preparation of future teachers plays an important role. Firstly, it enhances their mathematical culture and secondly, they develop skills and test-making algorithms, which are the most important components of their future professional activities.

3. Problems with non-traditional formulations of conditions

Most of the problems used in the process of mathematics teaching have a standard form: to solve the equation; to solve the inequality; to find the side of the triangle; to find the maximum point of the function, etc. We believe that such problems should mix from time to time by the problems of marginally unusual to the very unusual formulations.

If we do not do it in such situations the student is able to solve an equation with unknown *x*, but he is lost when instead of *x* in this equation is *t*; student, solving easily the equation $f(x)=g(x)$, cannot solve the problem “Find the abscissas of the intersection points of function graphs $y=f(x)$ and $y=g(x)$”, etc.

3.1. Problems with incomplete or redundant conditions

In formulating and solving of real-world problems we do not always have as much data as required. It may be more or it may be less data. That is why it is important to know how among all parameters of the problem to identify the significant and to dismiss the irrelevant. Therefore, the use of creating such problems in training is very useful. The following types of problems are offered:

a) If some constants are used in the problem (such as the Earth's radius, density, sound velocity, etc.), they are specified in the condition as a rule. We suggest not to always follow this method as the student must understand for himself independently what additional information he needs and then to find it in the literature, internet, etc.

b) If the problem is proposed for solution in the classroom, the teacher may deliberately omit some details. Students in the process of analyzing the problem and its solutions should ask the teacher specific questions as the ability to ask the right questions help to clarify the condition.

c) In default of data the students must consider several possible situations.

Example 1: What is $\sin x$, if $\cos x=\frac{3}{5}$?

The student must understand that the sign of the sine he could not identify, that is why it is necessary to consider two cases. Answer: $\frac{4}{5}$ when $x\in(2\pi n;\pi+2\pi n)$; $-\frac{4}{5}$ when $x\in(\pi+2\pi n;2\pi+2\pi n)$, $n\in Z$.

Example 2: Through the point, lying on the border of the trapezoid, draw a right line dividing the area of the trapezoid in half.

This difficult problem has different solutions, depending on where the condition point is defined. We have different situations: a) in one of the vertices of a trapezium; b) on the larger basis of trapezoid; c) on a smaller basis of trapezoid; d) on the lateral side. The student resolving this problem should understand it for himself.

It is noteworthy with geometric problems that more than one possible configuration can belong to the same type of problems.

d) The condition of the problem is really incomplete and there is no way to obtain the missing data. In this situation, the student must strictly prove non-solution of the problem.

Example: In the triangle the length of the base is 2 and the opposite angle is equal to $60°$. Find its area.

If we fix the base, the third vertex lies on the corresponding arc of a circle. Therefore, the height, lowered to the base of this peak can take infinitely many values. Hence, the area cannot be found. We can only determine that it varies from 0 to $\sqrt{3}$.

e) The condition of the problem is excessive. Therefore, a part of the conditions is used for solution of the problem. The other conditions serve as a check of the solution and accurate answers.

Example: In some two-digit number the first digit of 3 is greater than the second digit. Product of this number and the sum of digits is equal to 814. Find the number.

Not very simple analysis based on the fact that $814 = 2 \cdot 11 \cdot 37$ shows that the only number 74 satisfies the second condition of the problem. Therefore, the first condition can be used just to verify our solution; however the use of the solution of two conditions allows solving the problem by the majority of students.

f) The condition of the problem is excessive. For solution of the problem part of conditions is used while other conditions lead to a contradictory situation.

3.2. Problems with a contradictory condition

Formally, the problem is solved and an answer results. The behaviour of solutions is valid, but the answer for one or another reason cannot be correct. For example, it is obtained “1.5 diggers” like a bad pupil in one of the Russian cartoons or walking pace equal to 100 km/h.

Example: Sides of the parallelogram are equal to 7 and 5. The height drawn to the larger side is equal to 6. Find the second height of the parallelogram.

Formally obtained answer ($7 \cdot 6 = 5 \cdot x$; x = 8.4) is not suitable, because such parallelogram does not exist (the height of 6 cannot be greater than the side that is equal to 5).

3.3. Provocative problems

Provocative problems are such problems, the terms of which contain references and hints which push students to the choice of wrong solutions or wrong answers. Often it can be problems-traps or problems-tricks. They contribute to the education of critical thinking and they teach to analyse and evaluate information which increases the interest in learning mathematics.

Examples of such tasks:

1. The weight of a pencil is equal to 10 grams. Another pencil has double size measures. What is the weight of the second pencil? Answer: 80 grams, not 20 grams.
2. Which is the nearest prime number that follows 200? Answer: 211, not 201, 203 or 209.
3. Which of the numbers is more: a or $2a$? Answer: not known, it depends on the sign a .

3.4. Applied problems

This refers to problems in which precise mathematical statement of the condition is absent. The student must independently build a mathematical model, described in the condition of the situation, and only after this the student can solve the mathematical problem. Most often there are problems with the content of the natural sciences, economics and other applied problems. In the author's collection there are hundreds of such problems. Here are just a few examples.

In the study of line function instead of boring and a standard problem 1 (see below) it is much more interesting and more useful to give students the problem 2.

Problem 1. A line function $f(x) = ax + b$ is equal to 32 where $x = 0$ and is equal to 212 where $x = 100$. For what value of x the function is equal to 97.88?

System of equations $\begin{cases} 32 = a\cdot 0 + b, \\ 212 = a\cdot 100 + b \end{cases}$ leads to values $b=32$ and $a=\frac{212-32}{100}=\frac{9}{5}$, from which $f(x)=\frac{9}{5}x+32$. After this, the completion of the task does not cause difficulties.

Problem 2. In the USA, it is normal to calculate the temperature in Fahrenheit scale. On this scale, water freezes at *32F*, and it boils at *212F.* An American seventh-grader announced to his mother in the morning that he would not go to school because he had a temperature of *97.88 F.* Is this temperature high, normal or low?

In contrast to the previous problem, students must understand that there are two equivalent temperature scales and then conclude that they are connected by a line law: $C = aF + b$.

Further, from the conditions the constants *a* and *b* are determined and the formula of communication is the following: $C=\frac{5}{9}(F-32)$. By substituting in this formula the value *F* = 97.88 we find that the boy has a normal temperature of $36.6°C$.

In the study of irrational equations the problem 4 is preferred to the problem 3.

Problem 3. Solve the equation $\sqrt{ax}+bx=c$ for all positive values of the parameters *a, b* and *c.*

The equation $\sqrt{ax}=c-bx$ is equivalent to the system $\begin{cases} ax=(c-bx)^2, \\ c \ge bx. \end{cases}$ Equation of the system $b^2x^2-(a+2bc)x+c^2=0$ has two roots, only one of which $x=\frac{c}{b}+\frac{a-\sqrt{a^2+4abc}}{2b^2}$ satisfies to the inequality $x\le\frac{c}{b}$. For the root $x=\frac{c}{b}+\frac{a+\sqrt{a^2+4abc}}{2b^2}$ the condition $x\le\frac{c}{b}$ is not fulfilled.

Problem 4. An object was dropped from the tower without the initial velocity. Time from discharge to the receiving sound from the object when it hit the Earth's surface was made up of *t* seconds. Find the height of the tower *h* with the assumption that air resistance is negligible.

Denoting the speed of sound *V*, the gravitational acceleration *g*, we obtain: $S=\frac{gt^2}{2}$. In this case, a fall from a height *h* is equal to $t_1=\sqrt{\frac{2h}{g}}$. The propagation time of the sound from hitting the ground is equal to $t_2=\frac{h}{V}$. In accordance with the condition of the problem we obtain an irrational equation for the height of fall: $\sqrt{\frac{2h}{g}}+\frac{h}{V}=t$. It is the equation of the previous problem with $a=\frac{2}{g}$; $b=\frac{1}{V}$; $c=t$. It has a single root: $h=Vt-\frac{V^2}{g}\left(\sqrt{1+\frac{2tg}{V}}-1\right)$. It can be concluded from the formula of the previous problem.

Irrational equation in problem 4 is derived from the "real" life situation. It is important because often students consider irrational equations and inequalities as "useless". Also this problem illustrated which "extra root" corresponds to the irrational equation. Value $h=Vt+\frac{V^2}{g}\left(\sqrt{1+\frac{2tg}{V}}+1\right)$ "is not good" because it is larger than the path *Vt*, traversed by the sound of time *t.*

Enormous potential lies in applications that allow multiple ways of solution, especially if they include both the algebraic and geometric solution. Teacher "must deliberately simulate their students to solve problems in different ways to encourage the development of connected mathematical knowledge" (Leikin & Levav-Waynberg, 2007).

Problem 5. A ship is anchored at a distance of 200 meters from a straight coast and is preparing to sail. A late passenger is located at time 12:47 at a distance of 1400 meters from the ship as he runs along the waterfront. A) After which minimum possible time is it possible for the passenger to arrive at the dock of the ship, if he can swim with a speed of 4 km/h, and he can run by foot twice as fast? B) Does he have time to get to the ship, if the ship leaves at 13:00?

In this difficult task we can use some solutions: a) the algebraic solution, based on obtaining a function of time, and subsequently finding the minimum of this function by means of calculus; b) a very sophisticated geometrical solution; c) another geometric solution, based on the point Fermat-Torricelli-Steiner in a triangle; d) the physical solution is based on the laws of light refraction. A detailed analysis one of such problems provides students with an enormous advantage in the classroom.

4. Concluding Thoughts

Using these described techniques students will learn the material better and also understand the various nuances more profoundly. It is very important that these methods develops complete independence of students, their analytical thinking, self-control and critical attitude to the stated material.

For example, the theme "Irrational equations and inequalities" was repeated in two classes which were about the same level. The standard technique of repetition (an overview of the theory and solving of problems thereafter) was used in the first class. While in the second class repetition technique was based on the use of erroneous decisions. Control tests were conducted before repetition of the theme and after it. Figure 2 provides a comparison of the results.

a) Standard technique b) Described technique

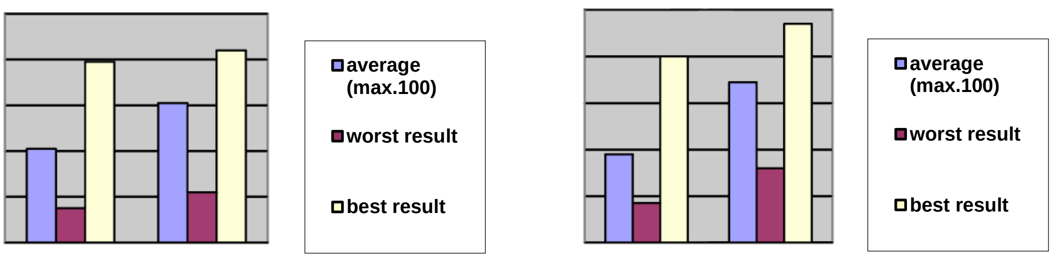

Figure 2. Comparison of the results of two methods

When we use described technique of repetition there is a significant increase over the average rating of the students. It is also important that the results of inferior students are significantly improved as well as the results of superior students.

Effectiveness of the described methods is proven by the high results of students on the National exams for mathematics. All of my students are in the top 10% of highest graduating results in Russia. Moreover half of them are in the top 1%. I would like to remind that these are the same students who two years before could not solve the problems like the irrational inequality $-\frac{1}{3}x < -2$.

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